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Nonlocal Symmetries and Associated Conservation Laws for Wave Equations with Variable Speeds

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We show that one can generate a class of nontrivial conservation laws for second-order partial differential equations using some recent results dealing with the *action* of any Lie–Bäklund symmetry generator of the equivalent first-order system on the respective conservation law. These conserved vectors are nonlocal as they are constructed from *associated* nonlocal symmetries of the partial differential equation. We demonstrate the complete procedure on certain classes of wave equations with variable wave speeds. Some of these have been considered in the literature using alternative methods.

1. INTRODUCTION

The generation of conservation laws of a system of differential equations from known ones using the symmetry properties of the system has been investigated over the years. In the case of ordinary differential equations, this result is well known as the "related integral theorem" and has found widespread application, for example, in classical mechanics (see, e.g., Sarlet and Cantrijn, 1981, and references therein). For a system of partial differential equations, a similar result has been established for canonical Lie–Bäcklund symmetries (see, e.g., Ibragimov, 1985). This result was extended to *any* Lie–Bäcklund symmetry without recourse to a Lagrangian formulation by Kara and Mahomed (2000).

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Anco and Bluman (1996) present a detailed discussion on the local nature of conservation laws derived from Noether's theorem. Nonlocal conservation laws are, by way of an identity, derived from nonlocal, canonical symmetries without reliance on a Lagrangian. In this article, we apply the identity derived by Kara and Mahomed (2000) to construct (nonlocal) conservation laws from nonlocal symmetries (not necessarily canonical) for some classes of wave equations with variable wave speeds. The nonlocal symmetries are, to be precise, potential symmetries which are a subclass of nonlocal symmetries.

To this end, we have the following definition with specific reference to the wave equation:

$$u_{tt} = c^2(x)u_{xx} \tag{1.1}$$

where $c(x)$ is the variable wave speed. Equation (1.1) can be written as the first-order system

$$u_t = c^2(x)v_x, \qquad u_x = v_t \tag{1.2}$$

Then any (Lie) symmetry vector $X = \xi\partial/\partial x + \tau\partial/\partial t + \eta\partial/\partial u + \phi\partial/\partial v$ of (1.2) is a nonlocal symmetry of (1.1) since v contains integrals of u (this latter statement is sometimes used as a formal definition for nonlocal symmetries). Bluman and Kumei (1986) give a detailed classification of the Lie point symmetries of (1.2) for some classes of $c(x)$. Also, Anco and Bluman (1996) present an account of nonlocal conservation laws of (1.1) constructed from a formula involving canonical symmetries.

We briefly outline the notation and give the necessary preliminaries.

Let $x = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$ be the independent variables with coordinates x^i, and let $u = (u^1, u^2, \ldots, u^m) \in \mathbb{R}^n$ be the dependent variables with coordinates u^α. The partial derivatives of u with respect to x are connected by the operator of total differentiation

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots, \qquad i = 1, \ldots, n$$

as

$$u_i^\alpha = D_i(u^\alpha), \qquad u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots$$

The collection of all first derivatives u_i^α will be denoted by $u_{(1)}$. Likewise, the collections of all higher order derivatives will be denoted by $u_{(2)}, u_{(3)}, \ldots$.

Consider an rth-order system of partial differential equations of n independent and m dependent variables,

$$E^\beta(x, u, u_{(1)}, \ldots, u_{(r)}) = 0, \qquad \beta = 1, \ldots, \tilde{m} \tag{1.3}$$

We recall that a *conserved form* of (1.3) is a differential $(n - 1)$-form

$$\omega = T^i(x, u, u_{(1)}, \dots, u_{(r-1)})\left(\frac{\partial}{\partial x^i} \rfloor (dx^1 \wedge \dots \wedge dx^n)\right) \tag{1.4}$$

if

$$\mathbf{D}\omega = 0 \tag{1.5}$$

is satisfied for all solutions of (1.3) (**D** is the operator of total exterior differentiation; see Anderson and Duchamp, 1984, or Kara and Mahomed, 2000, for a detailed analysis of total exterior differentiation).

Remark. When the above definition is satisfied, (1.5) is called a *conservation law* for (1.3).

It follows from (1.5) that

$$D_i T^i = 0 \tag{1.6}$$

on the solutions of (1.3), which is also referred to as a conservation law of (1.3). The tuple $T = (T^1, \dots, T^n)$ is called a *conserved vector* of (1.3).

We now review some definitions and results relating Lie–Bäcklund operators and possible conservation laws for systems which may not be derivable from a variational principle (see Ibragimov *et al.*, 1998, and references therein).

Suppose $\mathcal{A}$ is the universal space of differential functions. A Lie–Bäcklund operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \cdots \tag{1.7}$$

where $\xi^i, \eta^\alpha \in \mathcal{A}$ and the additional coefficients are

$$\begin{aligned} \zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ij}^\alpha \\ \zeta_{i_1 i_2}^\alpha &= D_{i_1} D_{i_2}(W^\alpha) + \xi^j u_{j i_1 i_2}^\alpha \\ &\cdots \end{aligned} \tag{1.8}$$

and W^α is the Lie characteristic function defined by

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha \tag{1.9}$$

The following theorem and definition are recalled from Kara and Mahomed (2000).

Theorem 1. Suppose that X is a Lie–Bäcklund symmetry of the system (1.3) such that the conserved form ω of (1.3), given by (1.4), is invariant under X. Then

$$X(T^i) + D_j(\xi^j)T^i - T^jD_j(\xi^i) = 0, \qquad i = 1, \ldots, n \tag{1.10}$$

Definition 1. A Lie–Bäcklund symmetry X is said to be *associated* with a conserved vector T (or its corresponding conserved form ω) of the system (1.3) if X and T satisfy (1.10).

2. APPLICATION TO THE WAVE EQUATION WITH VARIABLE WAVE SPEEDS

2.1. $c(x) = e^x$

We first consider the case $c = e^x$, so that (1.2) becomes

$$u_t = e^{2x}\, v_x, \qquad u_x = v_t \tag{2.1}$$

It is shown in Bluman and Kumei (1986) that a Lie point symmetry generator of (2.1) is

$$X = \frac{\partial}{\partial x} - t\,\frac{\partial}{\partial t} - v\,\frac{\partial}{\partial v}$$

which, consequently, is a nonlocal symmetry of (1.1). We use X to construct an associated (nonlocal) conservation law for (1.1) by applying the identity (1.10) and the conserved form (1.6) to (2.1). Equation (1.10) is the system

$$XT^1 + T^1D_x\xi - T^2D_x\tau = 0, \qquad XT^2 + T^2D_t\tau - T^1D_t\xi = 0 \tag{2.2}$$

where $\xi = 1$ and $\tau = -t$ and the conserved law (1.6) is

$$D_tT^1 + D_xT^2 = 0 \tag{2.3}$$

along the solutions of (2.1). As (2.1) is a first-order system, we will choose the conserved vector (T^1, T^2) to be independent of derivatives of u and v, i.e., dependent on x, t, u, and v. Thus, (2.2) becomes

$$\frac{\partial T^1}{\partial x} - t\,\frac{\partial T^1}{\partial t} - v\,\frac{\partial T^1}{\partial v} = 0, \qquad \frac{\partial T^2}{\partial x} - t\,\frac{\partial T^2}{\partial t} - v\,\frac{\partial T^2}{\partial v} = T^2 \tag{2.4}$$

which has the characteristic form

$$\frac{dx}{1} = \frac{dt}{-t} = \frac{dv}{-v} = \frac{dT^1}{0}, \qquad \frac{dx}{1} = \frac{dt}{-t} = \frac{dv}{-v} = \frac{dT^2}{T^2} \tag{2.5}$$

Both equations in (2.5) have invariants $c_1 = x + \ln t$, $c_2 = u$, and $c_3 = x + \ln v$, so that $T^1 = f_1(c_1, c_2, c_3)$ and $tT^2 = f_2(c_1, c_2, c_3)$.

The conserved form (2.3) in expanded form is

$$\frac{\partial T^1}{\partial t} + u_t \frac{\partial T^1}{\partial u} + v_t \frac{\partial T^1}{\partial v} + \frac{\partial T^2}{\partial x} + u_x \frac{\partial T^2}{\partial u} + v_x \frac{\partial T^2}{\partial v} = 0$$

along the solutions of (2.1), so that

$$\frac{\partial T^1}{\partial t} + e^{2x} v_x \frac{\partial T^1}{\partial u} + u_x \frac{\partial T^1}{\partial v} + \frac{\partial T^2}{\partial x} + u_x \frac{\partial T^2}{\partial u} + v_x \frac{\partial T^1}{\partial v} = 0 \tag{2.6}$$

Separating by derivatives of u and v, we have

$$\begin{aligned} u_x \quad &: \quad \frac{\partial T^1}{\partial v} + \frac{\partial T^2}{\partial u} = 0 \\ v_x \quad &: \quad e^{2x} \frac{\partial T^1}{\partial u} + \frac{\partial T^2}{\partial v} = 0 \\ &: \quad \frac{\partial T^1}{\partial t} + \frac{\partial T^2}{\partial x} = 0 \end{aligned} \tag{2.7}$$

In terms of invariants, (2.7) become

$$\begin{aligned} e^{c_1} \frac{\partial f_1}{\partial c_3} + e^{c_3} \frac{\partial f_2}{\partial c_2} &= 0 \\ e^{c_1 + c_3} \frac{\partial f_1}{\partial c_2} + \frac{\partial f_2}{\partial c_3} &= 0 \\ \frac{\partial f_1}{\partial c_1} + \frac{\partial f_2}{\partial c_1} + \frac{\partial f_2}{\partial c_3} &= 0 \end{aligned} \tag{2.8}$$

It is now a matter of solving (2.8); we utilize some ad hoc method here. Without delving into the details, it can be shown that a solution is given by

$$f_1 = f(\lambda, c_1) + g(\mu, c_1), \qquad f_2 = H(c_1) - e^{c_1}(g - f)$$

for some functions f, g, and H, where $\lambda = c_2 + e^{c_3}$ and $\mu = c_2 - e^{c_3}$. Also, the following equations are satisfied:

$$2 \frac{\partial g}{\partial c_1} + 2g + H' + H - D(\lambda, \mu) + \frac{1}{2} (\lambda - \mu)\left(\frac{\partial g}{\partial \mu} - \frac{\partial D}{\partial \lambda}\right) = 0$$

and $f + g = -H + D$.

We consider the following two cases.

2.1.1. $\partial g/\partial \mu = 0$, $H = c_1$

We obtain $D = L(\mu)/(\lambda - \mu)^2$, $g = -\frac{1}{2} c_1 + Ke^{-c_1}$, where L is some function of μ and K is constant. Thus, $f = -\frac{1}{2}c_1 - Ke^{-c_1} + L(\mu)/(\lambda -$

$\mu)^2$. Then, $f_1 = -c_1 + L(\mu)/(\lambda - \mu)^2$ and $f_2 = c_1 - e^{c_1}[2Ke^{-c_1} - L(\mu)/(\lambda - \mu)^2]$. The conserved vector (T^1, T^2) is then given by

$$T^1 = -x - \ln t + (2e^x v)^{-2} L(u - ve^x) \tag{2.9}$$

$$T^2 = \frac{1}{t}\left[x + \ln t - te^x\left(\frac{K}{e^x t} - \frac{L}{(2e^x v)^2}\right)\right]$$

We thus get

$$D_t T^1 + D_x T^2 = \frac{L'}{4v^2}\,[e^{-2x}u_t - v_x - e^{-x}(v_t - u_x)]$$

provided that

$$\frac{L}{2v^3}\,[e^{-x}v_x + e^{-2x}v_t] - \frac{L}{4e^x v^2} - \frac{L'}{4v} = 0 \tag{2.10}$$

and $L' \neq 0$. That is, $u_{tt} = e^{2x}u_{xx}$ has nonlocal symmetry X with associated nonlocal conservation law with components given by (2.9) subject to (2.10).

2.1.1. $g = \mu$, $H' + H = 0$ ($H = K\,e^{-c_1}$, K constant)

We now obtain $2\mu - D + \frac{1}{2}(\lambda - \mu)(1 - \partial D/\partial\lambda) = 0$, so that

$$D(\lambda - \mu)^2 = \frac{1}{3}\,\lambda^3 + \mu\lambda^2 - 3\mu^2\lambda + L(\mu)$$

for some function $L(\mu)$. Thus,

$$D = \frac{1}{12}\,[-c_2^3\,e^{-2c_3} + 7c_2^2\,e^{-c_3} + 5c_2 - 5e^{c_3}] + \frac{L}{4}\,e^{-2c_3} \tag{2.11}$$

$$g = c_2 - e^{c_3}, \qquad f = -K\,e^{-c_1} - c_2 + e^{c_3} + D$$

and

$$f_1 = -K\,e^{-c_1} + D$$

$$f_2 = K\,e^{-c_1} + e^{c_1}\,[K\,e^{-c_1} + 2c_2 - 2e^{c_3} - D]$$

Finally, the components of the conserved vector are

$$T^1 = -\frac{K}{te^x} + \frac{1}{12}\left[-\frac{u^3}{v^2\,e^{2x}} + 7\,\frac{u^2}{v\,e^x} + 5u - 5ve^x\right] + \frac{L(u - v\,e^x)}{4v^2\,e^{2x}}$$

and

$$T^2 = \frac{K}{t^2\,e^x} + e^x\biggl(\frac{K}{t\,e^x} + 2u - 2e^x\,v - \frac{1}{12}\left[-\frac{u^3}{v^2\,e^{2x}} + 7\,\frac{u^2}{ve^x} + 5u - 5v\,e^x\right] + \frac{L(u - v\,e^x)}{4v^2\,e^{2x}}\biggr)$$

The vector (T^1, T^2) is a nonlocal conserved vector associated with X for (1.1) if

$$L' + \frac{L}{v\,e^x}\left[\frac{2v_t}{v} + \frac{2v_x}{v} + 1\right] = 0$$

2.2. $c(x) = x$

We now consider the case $c(x) = x$; the corresponding first-order system becomes

$$v_t = x^2 v_x, \qquad u_x = v_t \tag{2.12}$$

The corresponding conservation law (1.6) is

$$\frac{\partial T^1}{\partial t} + x^2 v_x\,\frac{\partial T^1}{\partial u} + u_x\,\frac{\partial T^1}{\partial v} + \frac{\partial T^2}{\partial x} + u_x\,\frac{\partial T^2}{\partial u} + v_x\,\frac{\partial T^2}{\partial v} = 0 \tag{2.13}$$

which separates into

$$\begin{aligned} u_x \quad &: \quad \frac{\partial T^1}{\partial v} + \frac{\partial T^2}{\partial u} = 0 \\ v_x \quad &: \quad x^2\,\frac{\partial T^1}{\partial u} + \frac{\partial T^2}{\partial v} = 0 \\ &: \quad \frac{\partial T^1}{\partial t} + \frac{\partial T^2}{\partial x} = 0 \end{aligned} \tag{2.14}$$

We first find (T^1, T^2) associated with $X = x\,\partial/\partial x - v\,\partial/\partial v$, i.e., (1.10) becomes

$$x\,\frac{\partial T^1}{\partial x} - v\,\frac{\partial T^1}{\partial v} + T^1 = 0, \qquad \frac{\partial T^2}{\partial x} - v\,\frac{\partial T^2}{\partial v}$$

The characteristic form implies

$$T^1 = \frac{1}{x}\,f_1(c_1, c_2, c_3), \qquad T^2 = f_1(c_1, c_2, c_3) \tag{2.15}$$

where $c_1 = t$, $c_2 = u$, and $c_3 = xv$.

Differentiating (2.14a) and (2.14b) gives

$$\frac{\partial^2 T^1}{\partial v^2} = x^2 \frac{\partial^2 T^1}{\partial u^2}, \qquad \frac{\partial^2 T^2}{\partial v^2} = x^2 \frac{\partial^2 T^2}{\partial u^2}$$

and substituting (2.15), this pair becomes

$$\frac{\partial^2 f_1}{\partial c_3^2} = x \frac{\partial^2 f_1}{\partial c_2^2}, \qquad \frac{\partial^2 f_2}{\partial c_3^2} = x \frac{\partial^2 f_2}{\partial c_2^2}$$

As x cannot be written in terms of the c_i, we separate by powers of x, so that

$$f_1 = A(c_1)c_2 + B(c_1)c_3 + D(c_1), \qquad f_2 = P(c_1)c_2 + Q(c_1)c_3 + R(c_1)$$

Then

$$T^1 = \frac{1}{x}[A(t)u + B(t)xv + D(t)], \qquad T^2 = P(t)u + Q(t)xv + R(t)$$

and satisfying (2.14c) yields $A' = D' = 0$ and $B' + Q = 0$. Thus,

$$D_tT^1 + D_xT^2 = Au_t + Q(t)x^2v_x + x(B(t)v_t + P(t)u_x) + xv(B' + Q)$$

We may choose $Q = 1$. Thus, $A = -1$, $B = -t$, and $P = t$, so that

$$T^1 = -\frac{1}{x}(u + txv), \qquad T^2 = tu + xv$$

associated with $x\,\partial/\partial x - v\,\partial/\partial v$.

We present a summary of the calculations regarding the other symmetries. An analysis using the symmetry generator $u\,\partial/\partial u + v\,\partial/\partial v$ of (2.12) yields functions independent of v so that any possible conserved quantity is not nonlocal for $u_{tt} = x^2u_{xx}$.

Also, a linear combination of the above symmetries, viz., $x\partial/\partial x + \frac{1}{2}u\partial/\partial u - \frac{1}{2}v\partial/\partial v$, yields $T^1 = 1/x\, f_1(c_1, c_2, c_3)$ and $T^2 = f_1(c_1, c_2, c_3)$, where $c_1 = u^2/x$, $c_2 = t$, and $c_3 = xv^2$. Proceeding as above, it can be shown that f_1 and f_2 satisfy partial differential equations that yield

$$f_1 = 4A_1(c_2)\sqrt{c_3c_1} + 2D_1(c_2)\sqrt{c_3} + 2B_1(c_2)\sqrt{c_1} + E_1(c_2)$$
$$f_2 = 4A_2(c_2)\sqrt{c_3c_1} + 2D_2(c_2)\sqrt{c_3} + 2B_2(c_2)\sqrt{c_1} + E_2(c_2)$$

and therefore

$$T^1 = \frac{1}{x}\left[4A_1(t)vu + 2D_1(t)v\sqrt{x} + 2B_1(t)u\frac{1}{\sqrt{x}} + E_1(t)\right]$$

$$T^2 = 4A_2(t)vu + 2D_2(t)v\sqrt{x} + 2B_2(t)u\frac{1}{\sqrt{x}} + E_2(t)$$

subject to $D_2 + 2D_1' = 0$, $2B_1' - B_2 = 0$, and A_1, E_1 constants. The coefficients need to be satisfied after substitution into (1.6) along the solutions of (2.12).

Finally, the symmetry $2tx\,\partial/\partial x + 2\ln x\,\partial/\partial t + (tu - xv)\,\partial/\partial u - (tv + u/x)\partial/\partial v$ give rise to the associated symmetry condition on (T^1, T^2) given by

$$2tx\frac{\partial T^1}{\partial x} + 2\ln x\frac{\partial T^1}{\partial t} + (tu - xv)\frac{\partial T^1}{\partial u} - \left(tv + \frac{u}{x}\right)\frac{\partial T^1}{\partial v} + 2tT^1 - \frac{2}{x}T^2 = 0$$

$$2tx\frac{\partial T^2}{\partial x} + 2\ln x\frac{\partial T^2}{\partial t} + (tu - xv)\frac{\partial T^2}{\partial u} - \left(tv + \frac{u}{x}\right)\frac{\partial T^2}{\partial v} - 2xT^1 = 0$$

which is a coupled system and requires a more elaborate analysis than done above.

2.3. $c(x) = x^m$, $m \neq 0, 1$

Another interesting case, considered by Anco and Bluman (1996), is $c(x) = x^m$, $m \neq 0, 1$; we briefly consider the corresponding first-order system

$$u_t = x^{2m}v_x, \qquad u_x = v_t \tag{2.16}$$

whose conservation law (1.6) (after separating the 'coefficients' of u_x and v_x) gives rise to the system

$$\frac{\partial^2 T^1}{\partial v^2} = x^{2m}\frac{\partial^2 T^1}{\partial u^2}, \qquad \frac{\partial^2 T^2}{\partial v^2} = x^{2m}\frac{\partial^2 T^2}{\partial u^2}, \qquad \frac{\partial T^1}{\partial t} = \frac{\partial T^2}{\partial x} \tag{2.17}$$

The association condition (1.10) of (T^1, T^2) with $x\,\partial/\partial x + (1 - m)\,\partial/\partial t - mv\,\partial/\partial v$ reduces (2.17a), (2.17b) to

$$\frac{\partial^2 f_1}{\partial c_3^2} = \frac{\partial^2 f_1}{\partial c_2^2}, \qquad \frac{\partial^2 f_2}{\partial c_3^2} = \frac{\partial^2 f_2}{\partial c_2^2}$$

where $T^1 = (1/x)\,f_1(c_1, c_2, c_3)$, $T_2 = x^{m-1}f_2(c_1, c_2, c_3)$ and $c_1 = tx^{m-1}$, $c_2 = u$, $c_3 = vx^m$, so that

$$T^1 = \frac{1}{x}\,[A_1(u + vx^m, tx^{m-1}) + B_1(u - vx^m, tx^{m-1})]$$

$$T^2 = x^{m-1}[A_2(u + vx^m, tx^{m-1}) + B_2(u - vx^m, tx^{m-1})]$$

Substituting these forms into (2.17c) and solving provides the following additional form for T^2:

$$T^2 = x^{m-1}[P_2(tx^{m-1})(u + vx^m) + R_2(tx^{m-1})(u - vx^m) + S_2(tx^{m-1})]$$

subject to $\partial A_1/\partial c_1 = -\partial B_1/\partial c_1$. The final choices for A_1, B_1, P_2, R_2, and S_2 are made on substituting (T^1, T^2) into (1.6) along the solutions of (2.16).

3. REMARKS

We have shown that a class of conservation laws for wave equations with variable speeds can be constructed from nonlocal symmetries of the equation using some recent results concerning the *association* of symmetries with conservation laws. These are specific nonlocal symmetries, viz., potential symmetries, as these are Lie–Bäcklund symmetry generators of the equivalent system of first-order equations.

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